Interaction graphs of isomorphic Boolean networks

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Internship 2020/21

Title : Interaction graphs of isomorphic Boolean networks.

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Keywords : Boolean networks, directed graphs.

Context

A Boolean network (BN) with n components if a finite dynamical system described by the successive iterations of a function $f : \{0,1\}^n \to \{0,1\}^n$. BNs are classical models for the dynamics of real complex systems, such as gene and neural networks ; they also have many applications in Computer Science (network coding, memoryless computation).

The main parameter of f is its *interaction graph* : the vertices are the components, from 1 to n, and there is an arc from j to i if f_i depends on input j. A usual line of research consists in deducing some dynamical properties of f from its interaction graph only. This may have impacts in the context of gene networks : the first reliable experimental data obtained when a gene network is studied concern the interaction graph of this network (the interaction graph of f is well approximated, but f itself is unknown). So, denoting $F(G)$ the set of BNs whose interaction graph is G, the classical line of research consists in studying, according to G, the dynamical properties of the BNs in $F(G)$.

Detailed description of expected work

Here, we take, in some sense, the opposite direction. Let f, h be two Boolean networks with n components. They describe the *same* dynamics, up to an isomorphism, if there is a permutation π of $\{0,1\}^n$ such that $f \circ \pi = \pi \circ h$. However, even if f and h describe the same dynamics, the interaction graphs of f and h can be very different, and the aim of this internship consists in studying this phenomena (which, perhaps surprisingly, has not yet been studied). To initiate this study, a natural approach is to consider the set $\mathcal{G}(f)$ of the interaction graphs of the BNs isomorphic to f, and to study the size of $\mathcal{G}(f)$ according to f.

For instance, if f is the identity or a constant function, then $|\mathcal{G}(f)| = 1$. Are there other BNs for which this happens ? More generally, what kind of dynamical properties imply that $\mathcal{G}(f)$ is small ?

On the orther direction, can $\mathcal{G}(f)$ be very large ? If so, for which f ? In particular, we may think that, for each $\epsilon > 0$, if n is sufficiently large then $|\mathcal{G}(f)|/2^{n^2} \geq 1 - \epsilon$ for some f, that is, $\mathcal{G}(f)$ contains almost all the interaction graphs with n vertices. If true then it means that f is a kind of "universal dynamics" : for almost all interaction graphs G, some BN in $F(G)$ is isomorphic to f.

This subject is very open, since the set $\mathcal{G}(f)$ has not yet been studied. Many other questions are possible, and initiatives are widely encouraged.