Introduction to Finite Dynamical Systems

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1 The basic question

In many applications (mostly in biology) the interaction graph $G(f)$ of the system is known (or well approximated) while f itself is unknown. So the basic question is:

What can be said on the dynamics of f according to its interaction graph $G(f)$?

This is a difficult question since many different different networks f can have the same interaction graph (see Exercise 1). Given a graph G with vertex set $[n]$, we denote by $F(G)$ the set of Boolean networks $f : \{0,1\}^n \to \{0,1\}^n$ with an interaction graph $G(f)$ equal to G. The size of $|F(G)|$ is at least doubly exponential with the maximum in degree of G.

2 The acyclic case

To begin it is natural to make strong assumptions on the interaction graph. In the acyclic case, we obtain a rather clear situation: there is a global convergence toward a unique fixed point in at most n iterations.

Thorme 1 (Robert 1980)**.** *Let f be a finite dynamical system with n components. Suppose that $G(f)$ is acyclic, then f^n is a constant.*

We need some notations. Given a graph G and a vertex i inside, we denote by $N_G(i)$ the set of in-neighbors of i in G and we write $N(i)$ when G is given by the context. If $x \in \{0,1\}^n$ and $I \subseteq [n]$ then x_I is the restriction of x on I, that is, $x_I = (x_i)_{i \in I}$. the **Hamming distance** $d(x,y)$ between x and y is the number of $i \in [n]$ such that $x_i \neq y_i$.

Lemme 1. *For all $x, y \in \{0,1\}^n$, if $x_{N(i)} = y_{N(i)}$ then $f_i(x) = f_i(y)$.*

Proof. We proceed by induction on $d(x,y)$. If $d(x,y) = 0$ then $x = y$ and there is nothing to prove. Suppose $d(x,y) > 0$ and $x_{N(i)} = y_{N(i)}$. Then $x_k \neq y_k$ for some $k \in [n]$. Let x' with $x'_k = y_k$ and $x'_\ell = x_\ell$ for $\ell \neq k$. Since x and x' only differ in $x_k \neq y_k$, and since $k \notin N(i)$, we have $f_i(x) = f_i(x')$. Since $d(x',y) = d(x,y) - 1$, by induction hypothesis, we have $f_i(x') = f_i(y)$. Thus $f_i(x) = f_i(y)$. □

Proof of Theorem 1. Let $f : X \to X$ with $G(f)$ acyclic. Then $G(f)$ has a topological sort and we can assume, without loss, that for all $1 \leq i \leq j \leq n$ there is no edge from j to i in $G(f)$. If $x \in X$ and $t \geq 0$ then we set $x^t = f^t(x)$.

We prove the following by induction on i:

$$\text{for all } x, y \in X,\ i \in [n] \text{ and } t \geq i, \text{ we have } x_i^t = y_i^t \qquad (*)$$

Let $x, y \in X$. Since 1 is a source, f_1 is a constant function. Thus for $t \geq 1$ we have

$$x_1^t = f_1(x^{t-1}) = f_1(y^{t-1}) = y_1^t.$$

This proves the base case. For the induction step, let $2 \leq i \leq n$ and $t \geq i$. By the topological sort, we have $N(i) \subseteq \{1, \dots, i-1\}$ and thus, by induction hypothesis, $x_{N(i)}^{t-1} = y_{N(i)}^{t-1}$. We deduce from the lemma that

$$x_i^t = f_i(x^{t-1}) = f_i(y^{t-1}) = y_i^t$$

and this completes the induction step. In particular, we deduce from $(*)$ that

$$\textit{for all } x, y \in X,\ t \geq n, \textit{ we have } x^t = y^t. \tag{$**$}$$

We now prove that f has a fixed point. If not, then $\Gamma(f)$ has a limit cycle of length $\ell \geq 2$. If x and y are distinct states of this cycle, then, for all $k \geq 0$, we have

$$x^{k\ell} = f^{k\ell}(x) = x \neq y = f^{k\ell}(y) = y^{k\ell}$$

and we obtain a contradiction for $k\ell \geq n$. Thus f has a fixed point z (thus $z^t = z$ for all $t \geq 0$).

From $(**)$ we deduce that $x^n = z^n = z$, that is, $f^n(x) = z$ for all $x \in X$. □

3 Minimal and maximal number of fixed points

We are particularly interested in the connection between the interaction graph and the number of fixed points. For every graph G on $[n]$ we set

$$\min(G) := \min_{f \in F(G)} |\textsc{Fixe}(f)| \qquad \max(G) := \max_{f \in F(G)} |\textsc{Fixe}(f)|$$

By Robert's theorem, if $G(f)$ is acyclic then $\min(G) = \max(G) = 1$. We can prove the converse.

Thorme 2. *For every graph G, we have*

$$\min(G) = 0 \iff \max(G) \geq 2 \iff G \textit{ has a cycle.}$$

4 Exercises

1. *What is the size of $|F(C_n)|$?*

   Answer. If $f \in F(C_n)$ then each local transition function is either the copy of x_{i-1} or the negation of x_{i-1}, that is: $f_i(x) = x_{i-1}$ for all $x \in \{0,1\}^n$ or $f_i(x) = \overline{x_{i-1}}$ for all $x \in \{0,1\}^n$ (where x_0 means x_n). Thus we have two possible choices for each f_i, and thus 2^n for f. Hence, $|F(C_n)| = 2^n$.

2. *What are the sizes of $|F(K_3)|$ and $|F(K_4)|$?*

   Answer. Let $H(n)$ be the number of Boolean functions $h : \{0,1\}^n \to \{0,1\}$ that depends on its n inputs. We can give a recursive formula for $H(n)$. We have $H(0) = 2$ (two constant functions), $H(1) = 2$ (the copy and negation functions) and, more generally:

$$H(n) = 2^{2^n} - \sum_{i=0}^{n-1} \binom{n}{i} H(i).$$

For instance

$$H(2) = 2^{2^2} - \binom{2}{0} \cdot H(0) - \binom{2}{1} \cdot H(1) = 16 - 1 \cdot 2 - 2 \cdot 2 = 10$$

and

$$H(3) = 2^{2^3} - \binom{3}{0} \cdot H(0) - \binom{3}{1} \cdot H(1) - \binom{3}{2} H(2) = 256 - 1 \cdot 2 - 3 \cdot 2 - 3 \cdot 10 = 218$$

If $f \in F(K_n)$, each local function f_i depends exactly on $n-1$ inputs, and thus we have $H(n-1)$ possible choices. We deduce that $|F(K_n)| = H(n-1)^n$. In particular, $|F(K_3)| = 10^3$ and $|F(K_4)| = 218^4$.

3. *Prove that* $\max(C_n) \geq 2$ *and* $\min(C_n) = 0$*?*

   Answer. Let $f \in F(C_n)$ be defined by $f_1(x) = x_n$ and $f_i(x) = x_{i-1}$ for $2 \leq i \leq n$. In other words, $f(x) = (x_n, x_1, \ldots, x_{n-1})$. It is clear that if $x = (0, 0, \ldots, 0)$ (full-zero state) or $x = (1, 1, \ldots, 1)$ (full-one state) then $f(x) = x$. Thus $\max(G) \geq 2$. Now let $f \in F(C_n)$ be defined by $f_1(x) = \overline{x_n}$ and $f_i(x) = x_{i-1}$ for $2 \leq i \leq n$. In other words, $f(x) = (\overline{x_n}, x_1, \ldots, x_{n-1})$. Suppose that x is a fixed point. Then, $(x_1, x_2, \ldots, x_n) = (\overline{x_n}, x_1, \ldots, x_{n-1})$. Hence, $x_2 = x_1$, $x_3 = x_2$, $\ldots$, $x_n = x_{n-1}$ and we deduce that $x_1 = x_2 = x_3 = \cdots = x_n$. But then $x_1 = f_1(x) = \overline{x_n} = \overline{x_1}$, which is a contradiction. Thus f has no fixed points and we deduce that $\min(G) = 0$.