Introduction to Finite Dynamical Systems

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1 Basic definitions

A **Finite Dynamical System** with n components is a function $f : X \to X$ where $X = \prod_{i=1}^{n} X_i$ is a product of n finite integer intervals. We write

$$x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x)).$$

The function f is sometime called the **global transition function**, while its components f_i (which are functions from X to X_i) are called the local transition functions. The set X_i is the domain of evolution of the ith component; if $X_i = \{0,1\}$, we say that i is a *binary component*. The set X is the set of possible **states** (or **configurations**) for the system. At state $x \in X$, the state of the ith component of the system is $x_i \in X_i$. If $X = \{0,1\}^n$ (that is, every component is binary) we say that f is a **Boolean network** with n components. We will mainly study Boolean networks, since $X = \{0,1\}^n$ is enough to study many interesting phenomena. Boolean networks have many applications. In particular, in Biology, they are very classical models for gene and neural networks.

Example 1. *Here is an example of Boolean networks f with three components. The network f is defined under two equivalent ways: by giving the* table *of f, and by giving a definition of the local transition functions f_1, f_2, f_3 with* Boolean formula*.*

x	$f(x)$
000	000
001	110
010	101
011	110
100	001
101	100
110	101
111	100

$$\begin{cases} f_1(x) = x_2 \vee x_3 \\ f_2(x) = \overline{x_1} \wedge x_3 \\ f_3(x) = \overline{x_3} \wedge (x_1 \vee x_2) \end{cases}$$

1.1 Synchronous dynamics

In the most basic way, the dynamics of the system is described by the successives applications of f; this dynamics is called the ***synchronousdynamics***: if x^t is the state of the system at time t, then $f(x^t)$ is the system a time $t+1$. In other words, given an initial state x^0, the dynamics is described by the recurrence $x^{t+1} = f(x^t)$ for $t = 0, 1, \dots, n$. Thus, the system evolves in a **discrete time**. We denote by f^k the function that results from k compositions of f with itself:

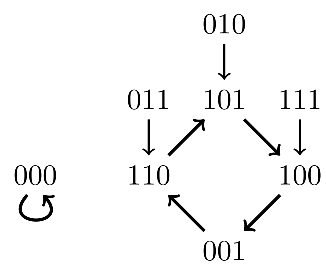
$$f^k := \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}.$$

Formally, f^0 is the identity on X and for every positive integer k, $f^{k+1} := f^k \circ f$. Hence, we have $x^{t+k} = f^k(x^t)$ for any integer t, k. In particular, $x^t = f^t(x^0)$.

A **periodic point** is a state x such that $f^k(x) = x$ for some $k > 0$; and the smallest such k is the **period** of x. A **fixed point** is a state x such that $f(x) = x$. Thus fixed points are exactly the periodic points of period one. We denote by $\textsc{Fixe}(f)$ the set of fixed points. A state which is not periodic is a **transient** state.

The synchronous dynamics can be visualized with the **synchronous transition graph** of f, denoted $STG(f)$. It is the directed graph with vertex set X and with an edge from x to $f(x)$ for all $x \in X$. The cycles of $STG(f)$ are called **limite cycles** or **attractors**. Clearly, a state is periodic if and only if it belongs to a limite cycle, and its period is then the length of this cycle. Note that limit cycles of length one in $STG(f)$ are exactly the fixed points of f.

Example 2. *Here is the $STG(f)$ of the Boolean network f given in Example 1.*



There are two limites cycles: one of length four, and one of length one. There are thus four periodic states of period four, and one periodic state of period one, that is, one fixed point. There are three transient states.

1.2 Asynchronous dynamics

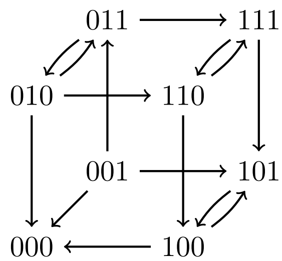
For each $i \in [n] = \{1, \dots, n\}$ we define $f^{(i)} : X \to X$ by

$$f^{(i)}(x) = (x_1, \dots, x_{i-1}, f_i(x), x_{i+1}, \dots, x_n).$$

Hence, $f^{(i)}(x)$ is the state obtained from x by updating the component i *only*. The **asynchronous transition graph** $ATG(f)$ described the possible evolutions of the system when, between each time step, only one component is updated: the vertex set is X and, for all $x \in X$ and $i \in [n]$, there is an arc from x to $f^{(i)}(x)$.

An ***attractor*** of $AST(f)$ is a smallest subset $A \subseteq X$ such that $ASG(f)$ has no arc $x \to y$ with $x \in A$ and $y \notin A$ (here "smallest" means "smallest with respect to the inclusion relation"). An attractor of size at least two is a **cyclic attractor**. Attractors of size one are **punctual attractors**. Note that punctual attractors in $ATG(f)$ are exactly the fixed points of f.

Example 3. *Here is the $ATG(f)$ of the Boolean network f given in Example 1. Loops (that is, cycles of length one) are not represented, since we can deduced it from the other arcs; for the* *example below, every vertex has a loop excepted* 001 *and* 010.

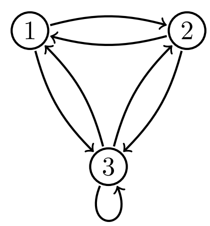


There are a unique attractor, namely $\{000\}$*, which is punctual and thus corresponds to a fixed point.*

1.3 Interaction graph

One of the most important parameter for f is its **interaction graph**, denoted $G(f)$. It is the directed graph with vertex set $[n] := \{1, \dots, n\}$ such that, for all $i, j \in [n]$, there is an edge from j to i if f_i depends on its jth input, that is, if there is $x, y \in X$ that only differ on $x_j \neq y_j$ such that $f_i(x) \neq f_i(y)$. If H is a directed graph on $[n]$, then $F(H)$ denotes the set of Boolean networks f with $G(f) = H$.

Example 4. *Here is the interaction graph of the Boolean network given in the first example.*



Remark 1. *In the following, graphs are always directed. Classical families of (interaction) graphs are* K_n *(the complete graph on* n *vertices: for distinct* $i, j \in [n]$ *there is an edge from* j *to* i *and from* i *to* j*),* C_n *(le cycle de longueur* n*: there is an edge from* i *to* $i+1$ *for* $1 \leq i < n$ *and from* n *to* 1*),* P_n *(le path with* n *vertices, of length* $n-1$*: there is an edge from* i *to* $i+1$ *for* $1 \leq i < n$*).*

2 The basic question

In many applications (mostly in biology) the interaction graph $G(f)$ of the system is known (or well approximated) while f itself is unknown. So the basic question is:

What can be said on the dynamics of f according to its interaction graph $G(f)$?

3 Exercises

1. *What is the number of n-component Boolean functions?*

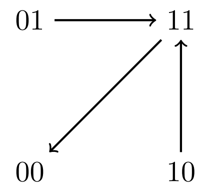
   Answer. The number of functions from a set A to a set B is $|B|^{|A|}$. Thus, the number of Boolean networks with n components is $(2^n)^{(2^n)} = 2^{n2^n}$ (doubly exponential with n).

2. *Find a Boolean network* $f : \{0,1\}^2 \to \{0,1\}^2$ *such that the number of attractors in* $ATG(f)$ *is* strictly *greater than the number of attractors in* $STG(f)$.

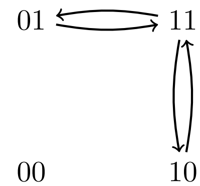
Answer. The sum is modulo two. Again, loops are not represented in $AST(f)$, and the loop on 00 in $STG(f)$ is not represented too.

x	$f(x)$
00	00
01	11
10	11
11	00

$$\begin{cases} f_1(x) = x_1 + x_2 \\ f_2(x) = x_1 + x_2 \end{cases}$$



$STG(f)$



$ATG(f)$

Hence, $STG(f)$ has a unique attractor, the fixed point 00, while $AST(f)$ has two attractors, one of size three, and the fixed point 00.