Problem Solving
Greedy Algorithm

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Notes

Greedy Algorithm

Definition

- A **greedy algorithm** is an algorithm that follows the principle of making a locally optimal choice at each step, with the hope of reaching a globally optimal result
- In cases where the algorithm does not always provide the optimal solution, it is called a **greedy heuristic**
- The method used to make the choice is sometimes referred to as a **greedy strategy**

Notes

Greedy Algorithm

Change with the Fewest Coins

Greedy strategy: At each step, return the coin with the largest value that is smaller than the remaining amount to be returned

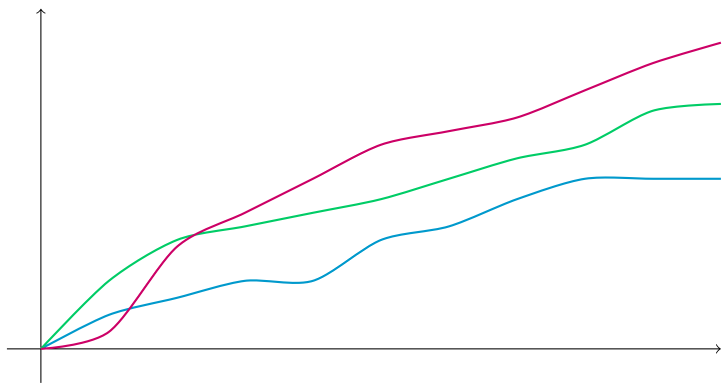
- To return 37 cents
- The largest coin is 20, so return 20, and 17 cents remain
- The largest coin is 10, so return 10, and 7 cents remain
- The largest coin is 5, so return 5, and 2 cents remain
- Return 2
- Thus, you have returned 20+10+5+2

- In the European coin system, the greedy algorithm always gives an optimal solution
- However, in the coin system (1, 3, 4), the greedy algorithm is not optimal (for 6: it gives 4+1+1, while 3+3 is optimal)

Notes

Greedy Algorithm

- Principle: be efficient
- Difficulty: finding the good strategy, or an efficient one



Notes

Knapsack Problem

Description

You have:

- A backpack with a weight limit
- A set of objects, each object o_i has:
  - A weight: p_i
  - A value: v_i

Which objects should be taken to maximize the total value carried while respecting the weight constraint?

- The total value of the selected objects is maximized
- The total weight of the selected objects is less than or equal to the backpack's weight limit

Notes

Knapsack Problem

You have bags of precious metals o_i and you can take as much as you want, but no more than p_i

For each bag, calculate the value per gram

1. Take the bag with the highest value per gram and fill the backpack with as much of that bag as possible
2. If the weight limit of the backpack is reached, stop
3. Otherwise, take the entire content of the bag, eliminate that bag, and repeat step 1

This strategy is optimal

Notes

Knapsack Problem

Optimality Proof

- Define the efficiency of an object o_i: as the value per gram
- Suppose there is an optimal solution that does not take one gram of an object o_i but does take one gram of an object o_k with lower efficiency than o_i
- By replacing 1g of o_k with 1g of o_i the solution improves

$\Rightarrow$ Contradiction

Notes

Knapsack Problem

- Putting bags of precious metal powder is equivalent to accepting cutting objects
- If we can't cut objects, what do we do?
- The problem becomes difficult
- We proceed "as if"
- We calculate the efficiency and select the objects based on their efficiency while respecting the weight constraint

Notes

Knapsack Problem

Example

We have a backpack with a maximum capacity of 15kg and the following items:

- o_1 of value 10 and weight 9kg
- o_2 of value 7 and weight 12kg
- o_3 of value 1 and weight 2kg
- o_4 of value 3 and weight 7kg
- o_5 of value 2 and weight 5kg

Notes

Knapsack Problem

Modelization

- How do we represent this problem?
- What are the variables (the unknowns)?
- How do we express the constraints?

This is the modelization: the mathematical representation of the problem

Notes

Knapsack Problem

Variables

- We associate each item with a 0-1 variable (it only takes values 0 or 1)
- It is a membership variable for the backpack
- If the item is taken, the variable is 1, otherwise it is 0

Model

- The value and weight of an item are given data, so for item o_i, we have the value v_i and the weight p_i
- The membership variable for the backpack is x_i
- The maximum weight of the backpack is W

Notes

Knapsack Problem

Constraints

- $max \sum_{i=1}^{n} v_i x_i$ the objective
- $\sum_{i=1}^{n} p_i x_i \leq W$ sum of weights less than or equal to the maximum weight

Notes

Heuristic

When the greedy algorithm does not always give an optimal solution, it is called a **heuristic method**

- It comes from the ancient Greek *eurisko*, meaning "I find"
- A heuristic is an algorithm that quickly (in polynomial time) provides a feasible solution, not necessarily optimal, for an NP-hard optimization problem
- A heuristic is an approximate method that does not always provide the exact solution

- We often refer to a heuristic method
- Generally, a heuristic is designed for a specific problem, relying on its particular structure

Notes

Evaluating a Heuristic

Practical or Empirical Criterion

The approximate algorithm is implemented, and the quality of its solutions is evaluated in comparison to optimal solutions (or the best-known solutions) on a *benchmark* (instances of the same problem accessible to all).

Mathematical Criterion

It must be shown that the heuristic guarantees performance, it is interesting to demonstrate a probabilistic guarantee when the heuristic often, but not always, provides good solutions.

Remarks

- Empirical and mathematical criteria can be contradictory
- Often, the guaranteed mathematical solution is not practical in real-life situations

Notes

Activity Selection Problem

Description

- We have a gym where multiple events take place: we want to fit in as many events as possible, knowing that two events cannot happen at the same time (there is only one gym)
- An event i is characterized by a start time s_i and an end time e_i
- Two events are compatible if their time intervals do not overlap

Question: How can we schedule the maximum number of events?

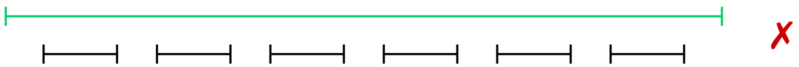
Notes

Activity Selection Problem

Strategy 1

Sort the events by increasing start time

Counterexample



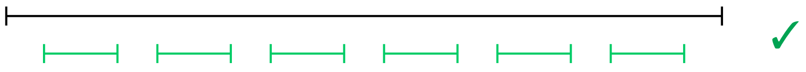
Notes

Activity Selection Problem

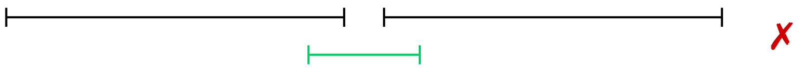
Strategy 2

Sort the events by increasing duration

Example



Counterexample



Notes

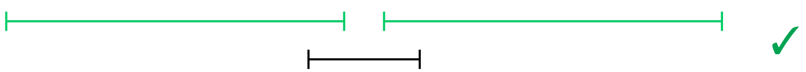
Activity Selection Problem

Strategy 3

Sort the events by increasing end time

Example

Example



Notes

Activity Selection Problem

Strategy 3

Sort the events by increasing end time

Property

This greedy algorithm is optimal

Notes

Activity Selection Problem

Proof by Induction

- Let f_1 be the earliest finishing event, let's show that there exists an optimal solution containing this event
- Let an arbitrary optimal solution be $O = f_{i1}, f_{i2}, ..., f_{ik}$ where k is the maximum number of events that can take place
- There are two possibilities:
  - either $f_{i1} = f_1$
  - or $f_{i1} \neq f_1$, we can replace f_{i1} with f_1 because it finishes before any other event, and since f_{i2} did not overlap with f_{i1}, f_{i2} will not overlap with f_1 either

    Therefore, we can find an optimal solution with f_1 as the first event
- Then, we only consider the events that do not overlap with f_1, and we repeat the process on the remaining events, hence the proof by induction

Notes

Exercise: Activity Selection

Description

- We have a gym where multiple events take place: we want to **maximize the gym's usage time**, knowing that two events cannot happen simultaneously (there is only one gym)
- An event i is characterized by a start time s_i and an end time e_i
- Two events are compatible if their time intervals do not overlap

Question: How can we maximize the gym's usage time?

Notes

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